10^4. Matched solutions were obtained at $R_{e_L} = 10^5$ with both matching procedures. Convergence was found monotonic with the new matching procedure and oscillatory (for the surface velocity) with the conventional matching procedure.[7]

Conclusions

A simple, flexible, and powerful method of matching solutions of different flow regions along a common boundary has been presented. Successful matching and rapid convergence to an accurately matched solution has been demonstrated for the problem of laminar flow over a flat plate. The method is obviously applicable to a much broader class of flow problems and their mathematical representations. It is apparent that the method can be applied to boundary-layer representations formulated in terms of integral relations or finite differences for either laminar or turbulent flows with appropriate turbulence modeling. Boundary layers, in turn, can be matched to any of several standard inviscid flow representations including numerical formulations such as linear influence coefficient methods, loading functions, or finite difference methods. Two examples of this kind, namely transonic small disturbance theory and finite difference representations of laminar boundary layers, are briefly discussed in Ref. 7.

Yet to be explored are the refinements that may be required when singular points occur in the matching, such as may appear as nonanalytic behavior in $dU/d\delta^*$ and/or $d\delta^*/dU$ in the vicinity of separation, shock/boundary-layer interaction, etc. In such cases it may be desirable to consider an additional imbedded region containing a mathematical description of the local phenomena and matched to the surrounding flow. An example of such a treatment for a problem involving separated flow is reported in Ref. 8.

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Vibration and Stability of Anti-Adjoint Elastic Systems

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STABILITY investigations of elastic systems lead to the eigenvalue analysis of homogeneous boundary-value problems. If the system is conservative in nature, the problem is self-adjoint, and if it is nonconservative in nature, the problem is nonself-adjoint. In this paper a special class of nonself-adjoint systems, namely, "anti-adjoint systems," is defined and the behavior of such systems studied.

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Definition

Consider an undamped, continuous, linearly elastic system occupying a domain V, inside a closed boundary S. Let the equation of motion be

$$\mu\, \partial^2 w/\partial t^2 + K[w] + PF[w] = 0 \tag{1}$$

where $w = w(x_j, t)$ = predominant deflection from the equilibrium position; x_j = spatial coordinates, t = time; $\mu = \mu(x_j)$ = mass density; P = force parameter; K = a self-adjoint operator in x_j, where $K[w]$ represents the stiffness; and F = a self-adjoint or nonself-adjoint operator in x_j, where $PF[w]$ represents the forces. Let the boundary conditions be

$$B[w] = 0 \quad \text{on } S \tag{2}$$

where B is a self-adjoint or nonself-adjoint operator in x_j.

Let the adjoint system be represented by the equation of motion

$$\mu\, \partial^2 u/\partial t^2 + K[u] + PF^*[u] = 0 \tag{3}$$

and the boundary conditions

$$B^*[u] = 0 \quad \text{on } S \tag{4}$$

where $u = u(x_j, t)$ = predominant deflection of the adjoint system, similar to w, and F^* and B^* are operators similar to F and B respectively.

If the system is self-adjoint

$$F^* \equiv F \quad \text{and} \quad B^* \equiv B \tag{5}$$

If at least one of these equalities does not hold, the system is nonself-adjoint. If

$$F^* \equiv -F \quad \text{and} \quad B^* \equiv B \tag{6}$$

such systems may be called "anti-adjoint systems."

Buckling Instability

Nonself-adjoint (nonconservative) systems can have two types of instability mechanisms—divergence (buckling) and flutter (oscillatory instability). Buckling is a static phenomenon, and at buckling the total potential energy is zero; i.e.,

$$\int_V K[w]\, w\, dV + P \int_V F[w]\, w\, dV = 0 \tag{7}$$

It is shown in the Appendix that for an anti-adjoint system

$$\int_V F[w]\, w\, dV = 0 \tag{8}$$

So, at buckling

$$\int_V K[w]\, w\, dV = 0 \tag{9}$$

However, since K represents the stiffness of the system, the Eq. (9) integral is positive definite and so the equation holds only for trivial solutions of w. Hence the anti-adjoint system can never buckle.

Reversal of Forces

Substitution of Eq. (6) in Eqs. (3) and (4) shows that the adjoint system of an anti-adjoint system can be obtained by merely reversing the direction of the forces in the original system (i.e., changing the sign of the force parameter in the equation of motion). Since the original and the adjoint systems have the same eigenvalues,[1] a reversal of the direction of the forces does not alter the frequencies of vibration of the system. Flutter boundaries of undamped, linearly elastic systems are characterized by the coalescence of two frequencies. Since the frequencies are unaltered, the flutter boundaries are also unaltered due to a reversal of the forces. The previous results may be summed up in two points: 1) anti-adjoint systems never buckle; and 2) frequencies and flutter boundaries of anti-adjoint systems are unaltered by a reversal in the direction of the forces.

Example

A typical example of an anti-adjoint system is given by the equation of motion

$$\mu(\partial^2 w/\partial t^2) + K[w] + P(\partial w/\partial x_j) = 0 \tag{10}$$

and the boundary condition

$$w = 0 \quad \text{on } S \tag{11}$$

This system, along with its application in aeroelasticity, has been discussed by the author in an earlier paper.[2] Where Ref. 2 was based on a two-term Galerkin approximation, the present work is mathematically exact.

Appendix

Let

$$w(x_j, t) = e^{i\omega t}\bar{w}(x_j) \tag{A1}$$

where ω is the frequency of vibration. Substituting Eq. (A1) in Eq. (1)

$$-\mu\omega^2\bar{w} + K[\bar{w}] + PF[\bar{w}] = 0 \tag{A2}$$

Since the original and the adjoint systems have the same frequencies,[1] let

$$u(x_j, t) = e^{i\omega t}\bar{u}(x_j) \tag{A3}$$

Substituting in Eq. (3)

$$-\mu\omega^2\bar{u} + K[\bar{u}] + PF^*[\bar{u}] = 0 \tag{A4}$$

By Green's identity,[3] the original and the adjoint operators are related by

$$\int_V \{-\mu\omega^2\phi + K[\phi] + PF[\phi]\}\psi \, dV$$
$$\equiv \int_V \{-\mu\omega^2\psi + K[\psi] + PF^*[\psi]\}\phi \, dV \tag{A5}$$

where $\phi(x_j)$ and $\psi(x_j)$ are functions satisfying the original and the adjoint boundary conditions, Eqs. (2) and (4), respectively. Since the boundary conditions are the same for an anti-adjoint system, and $w(x_j, t)$ satisfies these boundary conditions, let

$$\phi(x_j) \equiv w(x_j, t) \qquad \psi(x_j) \equiv w(x_j, t) \tag{A6}$$

Substituting in Eq. (A5) and simplifying

$$\int_V F[w] w \, dV = \int_V F^*[w] w \, dV \tag{A7}$$

Using Eq. (6) in Eq. (A7)

$$\int_V F[w] w \, dV = 0 \tag{A8}$$

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Gas-Particle Flow Past Bodies with Attached Shock Waves

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Introduction

THE problem of supersonic flow of a gas containing small particles (which may consist of water, ice, dust, or metal) past a solid body is of interest because the results obtained from its solution are pertinent to the estimation of the cratering and erosion damage experienced by a flight vehicle when it collides with such particles in the atmosphere. Some recent papers on this subject are those by Probstein and Fassio,[1] Waldman and Reinecke,[2] Spurk and Gerber,[3] Peddieson and Lyu,[4,5] and Lyu and Peddieson.[6] The purpose of the present Note is to put the governing equations for the dispersed phase of a dilute, low-mass-fraction, air-particle suspension (such as the atmosphere) in a form that is convenient for the numerical analysis of flows past symmetric bodies at zero angle of attack having attached shock waves. As an application of the equations, local collection efficiencies are computed for several body shapes and presented graphically.

Governing Equations

Figure 1 depicts the geometry of the problem and serves to define the axial coordinate x, the radial coordinate r, the gas velocity components u and v, the particle-phase velocity components u_p and v_p, the body surface $r_b(x)$, the shock surface $r_s(x)$ the body angle $\theta_b(x)$, the shock angle $\theta_s(x)$, the length of the region in which impacts are to be investigated L, and the freestream velocity U_∞, temperature T_∞, and particle-phase density $\rho_{p\infty}$. If the particle phase is treated as a continuum, body forces and interphase mass transfer are neglected, interphase momentum transfer is assumed to be linear in the difference between the velocity vectors of the two phases, and interphase heat transfer is assumed to be linear in the difference between the temperatures of the two phases (the last two assumptions are made to achieve the maximum simplicity in writing the equations and do not affect any of the subsequent analysis); the governing equations of the dispersed phase for steady axisymmetric flow can be written as

$$(r^j\rho_p u_p)_{,x} + (r^j\rho_p v_p)_{,r} = 0 \tag{1a}$$

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